

UW-IFT-19/96  
August, 1996  
hep-th/9609196

# Rigid string instantons are pseudo-holomorphic curves<sup>\*</sup>

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## Abstract

We show how to find explicit expressions for rigid string instantons for general 4-manifold  $M$ . It appears that they are pseudo-holomorphic curves in the twistor space of  $M$ . We present explicit formulae for  $M = R^4, S^4$ . We discuss their properties and speculate on relations to topology of 4-manifolds and the theory of Yang-Mills fields.

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<sup>\*</sup>Work supported in part by Polish State Committee for Scientific Research (KBN) and EC under the contract ERBCIP-DCT94-0034.

## 0 Introduction

Instantons play an important role in modern field theory and mathematics. Till now the thorough studies of instantons were carried for gauge fields and sigma models. Rigid string is another model of interest which was known to possess instantons. The model was originally considered as a string for gauge fields. Unfortunately, rigidity has quite a complicated structure when expressed in ordinary string variables. This prevents any significant progress in quantization. Certain rigid string instantons were derived and investigated in [1, 2]. Despite these efforts little was known about the generality of the proposed instanton equations and its significance for physics of the model.

In recent paper [3] we derived a new set of instanton equations for the 4d rigid string. It was claimed this set is rich enough to have representatives for all topological sectors of the rigid string. Because the action of the model contains terms with four derivatives the relevant topological invariant is not only the genus of the world-sheet surface but also the self-intersection number of the surface immersed in a target 4d space-time. The instantons split not into two families - instantons and anti-instantons but into three families. We shall call them  $J_1^{(P)}$ -instantons, anti- $J_1^{(P)}$ -instantons and minimal or  $J_2^{(P)}$ -instantons. Minimal instantons are just minimal maps from the world-sheet to the target space-time. In general, intersection of these families is non-trivial even in  $R^4$  what is also a novel feature. Unfortunately the equations seemed to be very difficult and a method (through the Gauss map) to solve them in full generality, failed.

In this paper we are going to study the rigid string instantons of [3] in more general setting. Thus we shall consider the rigid string moving in a Riemannian 4-manifold  $M$  with the metric  $G_{\mu\nu}^{(M)}$ . Using the twistor method [4, 5] we shall be able to show that in many cases one can give explicit formulas for the instantons. Moreover the construction will reveal an interesting structure of the equations, namely, the instantons will appear to be pseudo-holomorphic curves in the twistor space of  $M$ . This unexpected result unfolds the underlying simplicity of the equations and lays foundation of the successful solution of the equations.

It is worth to note that the subject touches Yang-Mills fields in two points. First of all, pseudo-holomorphic curves were used to build the string picture of  $YM_2$  [6]. Secondly the dimension of the moduli space of  $J_1^{(P)}$ -instantons on  $R^4$  and  $S^4$  is exactly

the same as those of  $SU(2)$  Yang-Mills instantons with the appropriate identification of topological numbers.

Content of the paper is the following: in Sec.1 we introduce the necessary notation and recall some results of [3]. In the next section we show how to solve the  $J_1^{(P)}$ -instanton equations using the twistor method. We also calculate the dimension of the moduli space of the rigid string instantons. In sec.3 we derive explicit formulas for the cases of  $M = R^4$  and  $M = S^4$ . In the final section we speculate on new topological (smooth) invariants of 4-manifolds. We also discuss connection of the rigid string instantons to string description of Yang-Mills fields and shortly discuss the case of 3-dimensional target  $M$ .

## 1 Rigid string instantons.

In this section we introduce necessary notions and recall basic results of [3].

We start with some generalities concerning the problem. We shall be interested in maps  $X : \Sigma \rightarrow M$  which are immersions i.e.  $rank(dX) = 2$  ( the tangent map is of maximal possible rank). Roughly speaking it means that the image of  $\Sigma$  in  $M$  is smooth. It means also that the induced metric  $g_{ab} \equiv \partial_a \vec{X} \partial_b \vec{X}$  is non-singular. Any immersion defines the Gauss map  $t^{\mu\nu} : \Sigma \rightarrow G_{4,2} = S_+^2 \times S_-^2$ . The appearance of product of two  $S^2$  corresponds to the fact that  $t^{\mu\nu}$  can be decomposed into self-dual  $t_+^{\mu\nu}$  and anti-self-dual  $t_-^{\mu\nu}$  part. If  $M$  has non-trivial topology we can not expect the Gauss map to be defined globally. Thus we must introduce the so-call Grassmann fiber bundle over  $M$  with fibers  $G_{4,2}$ . The map  $\tilde{X}$  to this bundle is called the Gauss lift. Because the fiber of this bundle splits into self-dual and anti-self-dual part we can consider Gauss lifts to each of them independently i.e. we can define bundle of tensors  $t_{mu}^{+ nu}$  separately. This is a sphere bundle which shall play a crucial role in the next section.

The action of the rigid string (without the Nambu-Goto term<sup>1</sup>) is

$$\int_{\Sigma} \sqrt{g} g^{ab} \nabla_a t^{\mu\nu} \nabla_b t_{\mu\nu} = 2 \int_{\Sigma} \sqrt{g} (\Delta X^{\mu}) (\Delta X^{\nu}) G_{\mu\nu}^{(M)} - 8\pi\chi. \quad (1)$$

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<sup>1</sup>The Nambu-Goto term breaks space-time scale invariance of the model thus prevents existence of instantons.

where  $t^{\mu\nu} \equiv \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu / \sqrt{g}$  are the element of the Grassmann manifold  $G_{4,2}$ ,  $G_{\mu\nu}^{(M)}$  is the metric on  $M$  and  $g_{ab} \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}^{(M)}$  is the induced metric on a Riemann surface of genus  $h$ . Tensors  $\partial_a X^\mu$  are components of  $T^*\Sigma \otimes X^*TM$ , where  $X^*TM$  is the pull-back bundle. The covariant derivatives are built with Levi-Civita connections on  $T^*\Sigma$  and  $TM$ . Explicitly  $\nabla_b \partial_a X^\mu = \partial_b \partial_a X^\mu - \Gamma_{ab}^{(\Sigma)c} \partial_c X^\mu + \Gamma_{\rho\sigma}^{(M)\mu} \partial_a X^\sigma \partial_b X^\rho$ . The Euler characteristic of the Riemann surface  $\Sigma$  is given by the Gauss-Bonnet formula  $\chi = \frac{1}{4\pi} \int_\Sigma \sqrt{g} R$ .

Immersion of Riemann surfaces in  $R^4$  are classified by the self-intersection number  $I$  [7]. General arguments based on singularity theory showed that rigidity separates topologically different string configurations. The derivation of instanton equations was based on the knowledge of relevant topological invariants. In our case these were the above mentioned self-intersection number  $I$  and the Euler characteristic  $\chi$ . The equations were derived using formulae for both invariants in terms of  $t^{\mu\nu}$ . Explicitly:  $\chi = I_+ - I_-$ ,  $I = \frac{1}{2}(I_+ + I_-)$ , where  $I_\pm = \pm \frac{1}{32\pi} \int_\Sigma \epsilon^{ab} \partial_a t_\pm^\mu \partial_b t_\pm^\nu t_{\pm\mu}^\rho t_{\pm\nu}^\rho$  and  $t_\pm^{\mu\nu} \equiv t^{\mu\nu} \pm \tilde{t}^{\mu\nu}$ . Standard reasoning yielded the following instanton equations (denoted as  $(+, \pm)$  with obvious sign convention)<sup>2</sup>:

$$-\nabla_a t^{+\nu}_\mu \pm \frac{\epsilon_a^b}{\sqrt{g}} t^{+\nu}_\rho \nabla_b t^{+\rho}_\mu = 0 \quad (2)$$

Here the equations were adopted to the general manifold  $M$ . Analogous equations hold for the anti-self-dual part of  $t^{\mu\nu}$ . The  $(+, +)$  equations are equivalent to  $\Delta X^\mu = 0$  and their solutions will be called minimal instantons or  $J_2^{(P)}$ -instantons. Appropriate equations for anti-self-dual part of  $t$  will give only  $J_1^{(P)}$ -anti-instantons - the fourth possibility appeared to be equivalent to the minimal instantons. Thus instantons form 3 families. The former two families behave as true instantons and anti-instantons in this sense that they do not have continuation to the Minkowski space-time and their role is interchanged under change of orientation of the space-time. Minimal instantons have continuation to Minkowski space-time what is a novel feature of this kind of solutions. It is also worth to note that change of orientation of the world-sheet (change of sign of the world-sheet complex structure) together with change of sign of  $t^{\mu\nu}$  (change of sign of the space-time complex structure) do not change any of the equations.

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<sup>2</sup>The minus – in front of the first term appeared in order to preserve notation of [3].

In  $R^4$  the instanton families are not disjoint. Intersection of  $J_1^{(P)}$ -instantons and minimal instantons gives  $\nabla_a t^{+\mu}{}_\nu = 0$  while intersection of anti- $J_1^{(P)}$ -instanton and minimal instantons gives  $\nabla_a t^{-\mu}{}_\nu = 0$ . These equations have solutions in  $R^4$  [2] and  $S^4$  [10, 4]. For  $R^4$  there is also one nontrivial intersection of  $J_1^{(P)}$ -instantons and anti- $J_1^{(P)}$ -instantons at genus zero. The solution was found in [3] to be a sphere embedded in  $R^3 \subset R^4$ . It has 5-dimensional moduli space - four positions and one breathing mode. The analogy with  $SU(2)$  Yang-Mills case is suggestive. In sec.2 we shall show that in fact the dimension of the moduli space of  $J_1^{(P)}$ -instantons on  $R^4$  and  $S^4$  is exactly given by the same formula as for the  $SU(2)$  Yang-Mills case with appropriate identification of topological numbers. We want to stress here that these properties of the three family of instantons were proven for  $M = R^4$  and may be modified for other 4-manifolds.

The above mentioned spherical solution was found using properties of the Gauss map of an immersion. Unfortunately we were not able to find other instantons with this method. In the following section we shall use the twistors [4, 5] in finding solutions to Eqs.(2). The method appears so powerful that one can find closed formulae for all instantons for many interesting spaces  $M$ .

## 2 Twistor construction of instantons

In this section we shall show that the rigid string instantons are pseudo-holomorphic curves in the twistor space of the space-time  $M$ . This will directly lead to the explicit formulas on instantons for some manifolds  $M$ . Moreover, the method will allow to calculate the dimension of the moduli space of instantons. In the following we shall concentrate on the self-dual part of  $t^{\mu\nu}$  only, understanding that the behavior of the anti-self-dual part is analogous.

Before we go to the main subject we recall some facts from complex geometry and twistors. We shall heavily use certain properties of almost complex structures.<sup>3</sup>

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<sup>3</sup>Several different almost complex structure will appear in this paper. In order to clarify the notation we decided to denote by  $\epsilon$  almost complex structures of 2d manifolds and by  $J$  almost complex structure of  $M$  and the twistor space  $\mathcal{P}_M$ . These will be supplemented with the appropriate superscript of the manifold. An almost complex manifold  $X$  with given almost complex structure  $J$  we shall denote as  $(X, J)$ .

Thus, the space of all almost complex structures on  $R^4$  is  $O(4)/U(2) = S^2 \times Z_2$  i.e. the space of all orthonormal frames up to unitary rotation which preserve choice of complex coordinates. The  $Z_2$  factor is responsible for change of orientation of  $R^4$ . Hence the bundle of almost complex structures over  $M$  (up to change of orientation) is just the sphere bundle  $\mathcal{P}_M$  i.e. a bundle with  $S^2$  as fibers. In other words, any point  $p \in \mathcal{P}_M$ , with coordinates in a local trivialization  $p = (u, x) \in S^2 \times R^4$ , fixes an almost complex structure  $J$  on  $M$  at  $x = \pi(p) \in M$ . This almost complex structure is given by the coordinate  $u$  on the fiber  $S^2$ .

It appears that such sphere bundles have two natural almost complex structures. The reason is that the sphere  $S^2$  has two canonical complex structures  $\pm\epsilon^{(S)}$ . In the conformal metric on  $S^2$  we have  $(\epsilon^{(S)})^{ij} = \epsilon^{ij}$ , where  $i, j = 1, 2$ . Out of  $\pm\epsilon^{(S)}$  we build two almost complex structures on  $\mathcal{P}_M$ . With the help of Levi-Civita connection on  $M$  we can decompose the tangent space  $T_p\mathcal{P}_M$  at  $p \in \mathcal{P}_M$  into the horizontal part  $H_p$  and the vertical part  $V_p$ :  $T_p\mathcal{P}_M = H_p \oplus V_p$ . The former is isomorphic to  $T_{\pi(p)}M$ . The isomorphism is given by the lift defined with help of the Levi-Civita connection on  $TM$ . The lift also defines the action of the almost complex structure  $J$  on  $H_p$ . The vertical space  $V_p$  is tangent to the fiber ( $S^2$ ) and has the complex structure  $\pm\epsilon^{(S)}$ . It follows that we can define two almost complex structure at  $p \in \mathcal{P}_M$  given by the formulas

$$\begin{aligned} J_1^{(P)} &= J \oplus \epsilon^{(S)} \\ J_2^{(P)} &= J \oplus -\epsilon^{(S)}. \end{aligned} \tag{3}$$

Both almost complex structures (3) will appear in the subsequent construction of the rigid string instantons. The sphere bundle  $\mathcal{P}_M$  with given almost complex structure  $J_1^{(P)}$  or  $J_2^{(P)}$  is sometimes called the twistor space [4] (see also [5]).

Now let us recall that a (complex) curve  $y$  from a Riemann surface  $(\Sigma, \epsilon^{(\Sigma)})$  to a manifold  $(N, J^{(N)})$  is said to be pseudo-holomorphic if

$$dy + J^{(N)} \circ dy \circ \epsilon^{(\Sigma)} = 0 \tag{4}$$

where  $dy$  is the tangent map  $dy : T\Sigma \rightarrow TN$ . Sometimes, in order to indicate the almost complex structure of the target space we shall call (4)  $J$ -holomorphic curve suppressing reference to  $\epsilon^{(\Sigma)}$  [9, 8]. In this paper we shall take  $\epsilon^{(\Sigma)}$  to be complex

structure given by  $(\epsilon^{(\Sigma)})_a{}^b = g_{ac}\epsilon^{cb}/\sqrt{g}$  where  $g$  is the metric on  $\Sigma$ . When we pull back the definition (4) on  $\Sigma$  we get

$$\partial_a y^m + (J^{(N)})_n{}^m \partial_b y^n \frac{\epsilon_a{}^b}{\sqrt{g}} = 0. \quad (5)$$

( $a, b, c = 1, 2 \dots m, n = 1, \dots \dim \mathcal{P}_M$ ). For the conformal metric and complex coordinates on  $\Sigma$  (5) is  $\bar{\partial}y^m - i(J^{(N)})_n{}^m \bar{\partial}y^n = 0$ . Thus  $\frac{1}{2}(1 - iJ^{(N)})$  is the projector on the holomorphic part, while  $\frac{1}{2}(1 + iJ^{(N)})$  on anti-holomorphic part of (complexified)  $TN$ .

As it was established in the previous section any immersions defines a sphere bundle. Explicitly we define  $\mathcal{P}_M$  as the bundle of normalized, self-dual tensors  $t_+^{\mu\nu}$  over  $M$ . The fiber of this bundle is homeomorphic to  $S^2$  (the normalization is  $t_+^{\mu\nu}t_{+\mu\nu} = 4$ ). The Gauss lift to this bundle will be denoted by  $\tilde{X}_+$ .

$$\begin{array}{ccc} & \mathcal{P}_M & \\ \tilde{X}_+ \nearrow & \downarrow \pi & \\ \Sigma & \xrightarrow{X} & M \end{array} \quad (6)$$

We see that this bundle is isomorphic to the bundle of almost complex structures defined previously. This is the reason why we used the same notation in both cases.

After establishing this simple fact we go to the instanton equation (2). We rewrite (2) and the equation which follows from definition of  $t_+$  in the conformal gauge for the induced metric  $g_{ab} \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}^{(M)} \propto \delta_{ab}$ .

$$\begin{aligned} (+, \pm) = \bar{\nabla} t_\mu{}^\nu \pm i t_\rho{}^\nu \bar{\nabla} t_\mu{}^\rho &= 0 \\ \bar{\partial} X^\mu - i t_\rho{}^\mu \bar{\partial} X^\rho &= 0 \end{aligned} \quad (7)$$

We have chosen complex coordinates on  $\Sigma$ , thus  $\bar{\nabla}$  is the anti-holomorphic part of the covariant derivative.

Next we show that Eqs.(7) give pseudo-holomorphic curves on  $\mathcal{P}_M$  with the two almost complex structures (3). Any Gauss lift defines  $t_+^{\mu\nu}$  and hence with the help of the metric  $G_{\rho\nu}^{(M)}$  we can write down an expression for the almost complex structure  $J_\mu{}^\nu = t_\mu{}^\nu$  on  $X^*TM$  at  $z \in \Sigma$ . We emphasize that  $J$  depends on coordinates on the Grassmann bundle  $\mathcal{P}_M$ . This almost complex structure decomposes (the complexification of) the tangent space  $X^*T_{X(z)}M$  into holomorphic  $T^{(1,0)}$  and

anti-holomorphic  $T^{(0,1)}$  part.<sup>4</sup> The former is defined as the space of vectors of the form  $T^{(1,0)} = \{(1-iJ)V; V \in X^*TM\}$  while the latter are complex conjugate vectors. We also choose locally almost hermitian metric which provides the following identification:  $T^{(0,1)} = T^{*(1,0)}$  and  $T^{*(0,1)} = T^{(1,0)}$ . Thus from tautology  $\frac{1}{2}(1-iJ)\frac{1}{2}(1-iJ)\frac{1}{2}(1-iJ) = \frac{1}{2}(1-iJ)$  we get  $\frac{1}{2}(1-iJ) \in T^{(1,0)} \otimes T^{*(1,0)}$  so  $J \in T^{(1,1)}$ . From  $J(\bar{\nabla}J) + (\bar{\nabla}J)J = 0$  we check that  $\frac{1}{2}(1-iJ)[(1-iJ)\bar{\nabla}J]\frac{1}{2}(1+iJ) = (1-iJ)\bar{\nabla}J$  i.e.  $(1-iJ)\bar{\nabla}J \in T^{(1,0)} \wedge T^{*(0,1)} \sim T^{(2,0)}$ . Similarly  $(1+iJ)\bar{\nabla}J \in T^{(0,2)}$ . Any self-dual tensors decomposes into direct sum  $T^{(2,0)} \oplus T^{(1,1)} \oplus T^{(0,2)}$  in the almost complex structure defined by  $J$ . This can easily checked in particular orthonormal basis of  $T^{(1,0)} \oplus T^{(0,1)}$ , e.g.  $\{e_1, e_2, \bar{e}_1, \bar{e}_2\}$ . In this basis  $J = e_1 \wedge \bar{e}_1 + e_2 \wedge \bar{e}_2$  and the two other self-dual tensors are  $e_1 \wedge e_2, \bar{e}_1 \wedge \bar{e}_2$ . We note that  $\bar{\nabla}J$  is also self-dual. Thus  $\bar{\nabla}J \in T^{(2,0)} \oplus T^{(0,2)}$ . As an immediate implication we infer that  $\bar{\nabla}J$  span the tangent space to the space of almost complex structures at the point  $J$ .

Using the above we can built two almost complex structure on the fibers  $S^2$ . We define  $\epsilon^{(S)}$  to be such an almost complex structure that  $T^{(2,0)}$  are holomorphic vectors while  $T^{(0,2)}$  are anti-holomorphic vectors. The choice  $-\epsilon^{(S)}$  would reverse holomorphicity properties. Thus,  $(1-iJ)\bar{\nabla}J$  is holomorphic, while  $(1+iJ)\bar{\nabla}J$  is anti-holomorphic in the  $\epsilon^{(S)}$  complex structure. One can easily find an explicit realization of  $\epsilon^{(S)}$  for  $M = R^4$ . For  $J_0^i \equiv n^i$ ,  $\vec{n} \in S^2$  and the following coordinate system on  $S^2$

$$\vec{n} = \left( \frac{f\bar{f}-1}{1+|f|^2}, -i\frac{f-\bar{f}}{1+|f|^2}, \frac{f+\bar{f}}{1+|f|^2} \right) \quad (8)$$

we get

$$(1-iJ)\bar{\partial}J = 0 \quad \Rightarrow \quad \bar{\partial}f = 0. \quad (9)$$

The above  $\epsilon^{(S)}$  is just standard complex structure on  $S^2$ . With the help of  $\pm\epsilon^{(S)}$  we can define two almost complex structures (3) on the fiber bundle  $\mathcal{P}_M$  just as we did in the beginning of this section.

Now it is easy to see that rigid string instantons (7) are pseudo-holomorphic curves  $\tilde{X}_+ : \Sigma \rightarrow (\mathcal{P}_M, J_{1,2}^{(P)})$ . Take  $J^{(P)}$  given by that of (3) and  $J, \epsilon^{(S)}$  defined as above.

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<sup>4</sup>We shall suppress the index  $X(z)$  of the tangent space at this point.



Hence if we split the map  $\tilde{X}_+$  into vertical and horizontal components of  $T\mathcal{P}_M$  then applying the notation of (3) we rewrite (4) as

$$(1 - iJ)dX = 0, \quad (1 \mp i\epsilon^{(S)})(d\tilde{X}_+)^v = 0 \quad (10)$$

In the first equation we have identified the horizontal component of the pseudo-holomorphic equation with its counterpart on  $M$ . In the second equation  $(d\tilde{X}_+)^v$  denotes the vertical part of the map i.e. the space of  $T^{(2,0)} \oplus T^{(0,2)}$  vectors. Thus, accordingly  $(1 \mp i\epsilon^{(S)})(d\tilde{X}_+)^v = (1 \mp iJ)\bar{\nabla}J$ . Recalling that  $J = t_+$ , this implies that (10) is equivalent to (7). We conclude that for conformal induced metric  $g_{ab} \sim \delta_{ab}$  on  $\Sigma$

*pseudo-holomorphic curves (4) are solutions of the instanton equations (7).*

The above considerations were applied in [4] in the context of minimal and conformal harmonic maps  $X : \Sigma \rightarrow M$ . In our present nomenclature these maps are  $J_2^{(P)}$ -holomorphic curves in  $\mathcal{P}_M$ . The almost complex structure  $J_2^{(P)}$  is non-integrable what makes pseudo-holomorphic curves on the manifold  $(\mathcal{P}_M, J_2^{(P)})$  hard to explore. We shall not dwell upon the case any more referring the reader to the existing reviews [11, 12].

On the other hand, the case of  $J_1^{(P)}$ -instantons maybe relatively easy. The reason is that in some cases the almost complex structure  $J_1^{(P)}$  is integrable thus defines a complex structure [14] on  $\mathcal{P}_M$ . There is a nice geometrical condition under which this happens. It states that  $M$  must be a half-conformally flat manifold [14, 4]. A lot of classical 4-manifolds respect this condition. In this work we shall concentrate on  $M = R^4, S^4$ . The other examples are  $T^4, S^1 \times S^3, CP^2, K3$ . Hence for the half-conformally flat  $M$  there exists complex coordinates  $\zeta_i$  on  $\mathcal{P}_M$  and then (4) is simply

$$\bar{\partial}\zeta_i = 0 \quad (11)$$

Thus  $J_1^{(P)}$ -instantons are just holomorphic maps  $\Sigma \rightarrow \mathcal{P}_M$ . Another important fact is that if  $J_1^{(P)}$  is integrable then it depends only on the conformal class of the metric  $G^{(M)}$  on  $M$ . This property gives  $J_1^{(P)}$ -instantons on  $R^4$  if they are known on  $S^4$  because  $R^4$  is conformally equivalent to  $S^4$ . The sphere bundle  $\mathcal{P}_M$  for the latter is  $CP^3$  with

unique complex structure being precisely  $J_1^{(P)}$ . Following this facts we shall construct all  $J_1^{(P)}$ -instantons for  $\Sigma = S^2$  explicitly in the next section.

There is a remark necessary at this point. We have chosen to work in the conformal metric  $g_{ab} = e^\phi \delta_{ab}$  on  $\Sigma$  thus fixing the almost complex structure on  $\Sigma$  from the very beginning. For higher genus surfaces Riemann surfaces  $\Sigma$  this is not possible globally unless one allows for some singularities of the metric i.e. vanishing of the conformal factor. In such a case solutions of the instanton equations will be so called branched immersions [4]. One may try to avoid this working with the most general complex structure  $\epsilon^{(\Sigma)}$ . This causes problems with the definition of almost complex structures on  $\mathcal{P}_M$ . It is because, for the rigid string,  $\epsilon^{(\Sigma)}$  is determined by the induced metric from  $X$ , but not from  $\tilde{X}$ . The problem can be resolved if both metrics are the same what happens for intersection of  $J_1^{(P)}$  and  $J_2^{(P)}$  families. It appears that if  $M = S^4$  then all minimal surfaces respect this condition [10].

## 2.1 Moduli space

We define the moduli space  $\mathcal{M}$  of the problem (7) as the space of solutions modulo automorphism group of solutions and reparameterizations of  $\Sigma$ . This moduli space is the same as the moduli space of (4). One of interesting quantities is the dimension of  $\mathcal{M}$ . Unfortunately, fixing the metric on  $\Sigma$  to be conformal we have lost control (except the case when  $\Sigma = S^2$ ) over the space of reparameterizations. Thus we first calculate the dimension of the space  $\tilde{\mathcal{M}}$  of solutions of (4) with fixed metric and then we shall argue how to correct formula in order to get  $\dim(\mathcal{M})$ .

The (virtual) dimension of the moduli space  $\dim \tilde{\mathcal{M}}$  is expressed through an index of an operator [15] The latter is a deformation of (4):  $\tilde{X}_+ + \xi : \Sigma \rightarrow \mathcal{P}_M$ . After short calculations we get the deformation of (4):

$$[(1 - iJ^{(P)})\bar{\nabla}\xi - i\bar{\nabla}_\xi J^{(P)}] \equiv (1 - iJ^{(P)})\bar{\nabla}\xi + O(\xi) = 0. \quad (12)$$

where  $O(\xi)$  denotes terms linear in  $\xi$  and not containing derivatives of  $\xi$ . The operator in (12) acting on  $\xi$  is the elliptic (twisted) operator mapping  $\tilde{X}_+^* T\mathcal{P}_M \rightarrow \Lambda^{(0,1)}\Sigma \otimes \tilde{X}_+^* T\mathcal{P}_M$ . Homotopic deformations of the  $O(\xi)$  part does not change its index [15, 8]. Thus we can set it to zero and obtain the Dolbeault operator  $\bar{\partial}_J = (1 - iJ^{(P)})\bar{\partial}$ .

The index is given by general Atiyah-Singer theorem or by Hirzerbruch-Riemann-Roch theorem.

$$\begin{aligned}\text{Index}(\bar{\partial}_J) &= c_1(\tilde{X}_+^* T\mathcal{P}_M) + \frac{1}{2} \dim_C(\mathcal{P}_M) c_1(T\Sigma) \\ &= c_1(\tilde{X}_+^* T\mathcal{P}_M) + 3(1-h).\end{aligned}\tag{13}$$

Thus  $\dim_R(\tilde{\mathcal{M}}) = 2c_1(\tilde{X}_+^* T\mathcal{P}_M) + 6(1-h)$ . For  $g = 0$  the moduli space  $\mathcal{M}$  is  $\tilde{\mathcal{M}}$  divided by the action of the group of automorphisms of  $S^2$  i.e. the Möbius group. Hence we obtain  $\dim_R(\mathcal{M}) = 2c_1(\tilde{X}_+^* T\mathcal{P}_M)$ . For higher genus surfaces  $h > 0$  if one assumes that the metric on  $\Sigma_h$  is elementary or induced from  $\mathcal{P}_M$  one would get

$$\dim_R(\mathcal{M}) = \dim_R(\tilde{\mathcal{M}}) - 6(1-h) = 2c_1(\tilde{X}_+^* T\mathcal{P}_M).\tag{14}$$

The result agrees with [9] where  $\mathcal{M}$  denotes the space of unparameterized pseudo-holomorphic curves  $\Sigma \rightarrow \mathcal{P}_M$ . It is interesting to notice that the formal expression on  $\dim_R(\mathcal{M})$  is independent on the almost complex structure on  $\mathcal{P}_M$ . Thus one can use the same formula for both families of instantons [9, 6].

It is known that for  $M = S^4$  the sphere bundle is  $CP^3$ . In this case we can easily find the dimension of  $\mathcal{M}$  for maps from  $\Sigma = S^2$ . If the map  $S^2 \rightarrow CP^3$  is given by the degree  $k$  polynomials in the variable  $z$  we get  $\dim_R(\mathcal{M}) = 2c_1(\tilde{X}_+^* CP^3) = 2kc_1(CP^3) = 8k$ .

## 3 Explicit formulae

### 3.1 $M = S^4$

From now on we shall discuss explicit solutions of the  $J_1^{(P)}$ -instanton equations. There is vast literature for the minimal instanton case [4] and we are not going to review it here.

It is known that for  $S^4$  the appropriate twistor space is  $CP^3$  which has only one complex structure. Complex projective space  $CP^3$  is defined as projective subspace of  $C^4$  i.e.  $CP^3 = C^4 / \sim$  where  $\sim$  means that we identify  $(Z_1, Z_2, Z_3, Z_4)$  and  $\lambda(Z_1, Z_2, Z_3, Z_4)$  for all  $0 \neq \lambda \in C$ . We can cover  $CP^3$  with four charts  $k = 1, \dots, 4$  for which  $Z_k \neq 0$  respectively. In the  $k$ -th chart we introduce (inhomogeneous) coordinates:  $\zeta_i \equiv Z_i/Z_k$  ( $i \neq k$ ). Eq. (11) implies that  $\zeta_i$  are meromorphic functions of  $z$  on  $\Sigma$ . This yields instantons on  $\mathcal{P}_M$  which next must be projected on  $S^4$ . We do this with help of a very

convenient representation of  $S^4$  as the quaternionic projective space [16]. We recall that quaternions are defined as  $q = q^m \sigma^m$  ( $m=0,..3$ ),  $\sigma^m = (1, i, j, k) \equiv (1, i\vec{\sigma})^5$ . The space of quaternions is denoted by  $H$  and is isomorphic to  $C^2$ . The isomorphism is such that  $(Z_1, Z_2, Z_3, Z_4) \leftrightarrow (Z_1 + jZ_2, Z_3 + jZ_4) \in H^2$ . Multiplication and conjugation of quaternions follows from the above matrix representation. Now we have

$$S^4 = HP^1 \equiv H^2 / \sim \quad (15)$$

In the above  $\sim$  means that we identify  $(q_1, q_2)$  and  $(q_1 q, q_2 q)$  for all  $0 \neq q \in H$  i.e.  $S^4$  is quaternionic projective space (line). Quaternionic representation of  $S^4$  is so useful because  $CP^3$  is complex projective space in the same  $C^4$ . Heaving a curve in  $CP^3$  we can represent it in  $H^2 = C^4$  and then define two maps  $H^2 \rightarrow R^4$  which cover  $S^4$ :  $(q_1, q_2) \rightarrow (q_1, X_+ q_1)$  for  $|q_1| \neq 0$ , and  $(q_1, q_2) \rightarrow (X_- q_2, q_2)$  for  $|q_2| \neq 0$ . The maps are stereographic projections of  $S^4$  from the north and south poles with the transition function  $X_- = 1/X_+$ . The norm is  $|X|^2 = (X^\dagger X) = XX^\dagger$  (the expression is proportional to the unit matrix). Explicitly we have

$$X_+ = (Z_3 + jZ_4)(Z_1 + jZ_2)^{-1} = \frac{(\bar{Z}_1 Z_3 + Z_2 \bar{Z}_4) + j(\bar{Z}_1 Z_4 - Z_2 \bar{Z}_3)}{|Z_1|^2 + |Z_2|^2} \quad (16)$$

Rotations  $SO(4) = SU_L(2) \times SU_R(2)/Z_2$  act as  $X'_+ = (\alpha_L + j\beta_L)X_+(\alpha_R + j\beta_R)$ . We see that the action of both  $SU(2)$  groups (here unit quaternions) is equivalent.

After these general remarks we go to the detailed description of the  $J_1^{(P)}$ -instantons with topology of sphere  $S^2$ . Let us first reproduce the only compact  $J_1^{(P)}$ -instanton found in [3]. We take  $Z_i = a_i(z + b_i)$  ( $i=1,..4$ ) i.e. a complex line in  $C^4$ . For generic choice of  $\{a_1, a_2, b_1, b_2\}$  the quaternion  $q_1$  is not singular  $q_1 = Z_1 + jZ_2 \neq 0$ . By the conformal transformation (Möbius group),  $z \rightarrow \frac{\alpha z + \beta}{\gamma z + \delta}$  ( $\alpha, \beta, \gamma, \delta \in C$ ,  $\alpha\delta - \beta\gamma = 1$ ), we can fix position of 3 point. Thus we choose  $b_1 = \infty, b_2 = 0, a_1 = a_2$ . Going from  $C^4$  to  $CP^3$  fixes  $a_1 = a_2 = 1$  so  $(Z_1 + jZ_2) = (1 + jz)$ . Then we get

$$X_+ = \frac{(Z_3 + jZ_4)(1 - \bar{z}j)}{1 + |z|^2} = X_0 + \frac{(Az + B) + j(-\bar{B} + \bar{A})}{1 + |z|^2} \quad (17)$$

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<sup>5</sup>According to the standard notation,  $i$  on the l.h.s. of this definition denotes the matrix, while on the r.h.s., the imaginary unit. This remark is applicable whenever we use quaternions.

for some constants  $X_0 \in H$ ,  $A, B \in C$ . Moding out by the rotation group leaves only the scale  $\lambda$  and the position  $X_0$  as moduli . Hence

$$X - X_0 = \frac{\lambda}{1 + |z|^2}(z + j), \quad \lambda \in R \quad (18)$$

what is exactly the result obtained in [3]. (18) represents sphere of radius  $\lambda/2$ . The above shows that (18) is the most general  $J_1^{(P)}$ -instanton with  $\chi = 2, I = 0$ .

We can easily generalize this to other topological sectors. In order to get  $J_1^{(P)}$ -instantons of the  $k$ -th sector the functions  $Z_i$  which defines  $\zeta_i$  must be polynomials of degree  $k$

$$Z_i = a_i \prod_{j=1}^k (z - a_{ij}) \quad i = 1, \dots, 4 \quad (19)$$

Thus  $\zeta_i$ 's are rational functions with poles at points where coordinates are ill defined. We can calculate dimension of the moduli space  $\mathcal{M}$  directly from (16,19). There are  $8k+6$  parameters involved in (16). Moding out by the Möbius group subtract 6 parameters yielding  $\dim(\mathcal{M}) = 8k$ . We can also divide by the rotation group dropping additional 3 dimensions of the moduli space. The instanton sectors are characterized by the self-intersection number of the immersed surface in  $S^4$ :  $I = k - 1$ . We shall obtain this result by simple means in the next subsection. The dimension of the moduli space is quite remarkable result, because it is exactly the dimension of the moduli space of  $SU(2)$  instantons [16]. Moreover we for  $k = 1$  topology of both spaces is exactly the same. Topology of  $\mathcal{M}$  for higher  $k$  remains to be investigated.

### 3.2 $M = R^4$

It appeared that the rigid string instanton equations, which seemed so complicated [3], can be trivially solved in  $R^4$ . Using the parameterization of (8) we can rewrite the second of Eqs.(7) as:

$$\begin{aligned} \bar{\partial} \bar{X}_+^1 + f \bar{\partial} X_+^2 &= 0 \\ -f \bar{\partial} X_+^1 + \bar{\partial} \bar{X}_+^2 &= 0 \end{aligned} \quad (20)$$

where  $X_+^1 = X^0 + iX^1$ ,  $X_+^2 = X^2 + iX^3$ . This is enormous and unexpected simplification of the  $(1 - it_+)\bar{\partial}X = 0$  equation. The first line of Eqs.(7) is

$$\bar{\partial}f = 0 \quad \text{for } J_1^{(P)}\text{-instantons} \quad (21)$$

$$\bar{\partial}\bar{f} = 0 \quad \text{for minimal instantons} \quad (22)$$

Both system of equations are very simple and can be directly integrated.  $J_1^{(P)}$ -instantons are identical with (17). Explicitly

$$X_+^1 = \frac{\bar{w}_1(\bar{z}) - \bar{f}(\bar{z})w_2(z)}{1 + |f|^2}, \quad X_+^2 = \frac{\bar{w}_2(\bar{z}) + \bar{f}(\bar{z})w_1(z)}{1 + |f|^2} \quad (23)$$

Comparing with (16) we see that  $f = Z_2/Z_1$ . Because  $I_+$  is minus degree of the map:  $f : \Sigma \rightarrow S^2$  we get  $I_+ = k$ . From the relation:  $I_+ = I + \chi/2$  follows that  $I = k - 1$ , what is the result quoted in the previous subsection.

We want to stress that results on  $M = R^4$  and  $M = S^4$  are almost identical because  $S^4$  is conformally equivalent to  $R^4$  and the integrable  $J_1$  almost complex structure is conformally invariant [14]. We also notice non-triviality of the complex structure given by  $f = f(z)$ : holomorphic functions are  $\bar{X}_+^1 + f X_+^2$  and  $-f X_+^1 + \bar{X}_+^2$ .

Minimal instantons can also be integrated and as one could expect they give solutions of the equation  $\Delta X^\mu = 0$ . Contrary to the previous case they do not correspond to minimal surfaces on  $S^4$ . We shall not dwell upon this subject referring to the rich existing literature [4, 10, 12].

## 4 Speculations and final remarks

In this section we allude on some possible applications of the presented results to topology of 4-manifolds and indicate similarities with several proposals for string picture of gauge fields. We also shortly discuss the case of 3d target manifold.

### 4.1 Topology of 4-manifolds

Starting from works of Gromov [9] and Witten [13] pseudo-holomorphic curves were used to define certain topological invariants, so called Gromov-Witten invariants [8] of symplectic manifolds (here denoted by  $N$ ). The invariants can be defined geometrically in descriptive way as follows: take a set of homology cycles  $\alpha_i \in H_{d_i}(N, \mathbb{Z})$  and

count (with an appropriate sign) those pseudo-holomorphic curves representing 2-cycle  $A \in H_2(N, \mathbb{Z})$  which intersect all classes  $\alpha_i$  at some points. There is also “physicist” definition of the invariants through a correlation function in a topological field theory [13]. In this case the invariants can be formally defined on any almost complex manifold.

All of twistor spaces are almost complex and some of them are Kähler (for  $M = S^4, CP^2$ ) so also symplectic. Thus following these definitions one could define appropriate invariants for the twistor spaces of 4-manifolds  $M$  considered in this work. The hypothesis is: *the Gromov-Witten invariants of the twistor space  $\mathcal{P}_M$  define some invariants of the 4-manifold  $M$ .* These new invariants are well defined on  $M$  if they are well defined on  $\mathcal{P}_M$ . Moreover we can define two sets of invariants (if we require that  $\mathcal{P}_M$  must be almost complex only) due to two natural almost complex structures  $J_{1,2}^{(P)}$  on  $\mathcal{P}_M$ .

The real problem is what kind of topological information do they carry? Intersection of cycles in the twistor space (say at  $p \in \mathcal{P}_M$ ) corresponds to the situation when projection of the cycles to  $M$  have common tangents at common point  $\pi(p) \in M$ . This property is invariant only under diffeomorphisms of  $M$  (class  $C^1(M)$ ) but not under homeomorphisms of  $M$ !. It may be that the invariants carry some information about smooth structures of  $M$ , so would be similar in nature to Donaldson or Seiber-Witten invariants. The basic difference is that they are defined in purely geometrical way avoiding any reference to gauge fields. Moreover the invariants seems to be well defined on manifolds for which there are no other invariants. This includes very interesting cases  $M = R^4, S^4$  discussed in this paper. Both cases are, of course, different because there are no compact  $J_2^{(P)}$ -instantons on  $R^4$ . Contrary, the  $J_1^{(P)}$ -invariants should be the same due to one-to-one correspondence between spaces of instantons in both cases. This subject, if relevant, seems to be very exciting.

## 4.2 Relation to gauge fields

Going back to physics we want to discuss striking relations of rigid string with gauge fields. Of course both theories uses twistors in construction of instantons. Leaving this aside we go to more quantitative comparisons. First of all, two-dimensional pseudo-holomorphic curves were used to build the string picture of  $YM_2$  [6]. Rigid string

instantons provides natural generalization of these curves to 4-dimensions. One can perform a naive dimensional reduction of 4d instantons to 2-dimensions by suppressing two coordinates (say  $X^2, X^3$ ). This results in taking  $|t^{01}| = 1$  (there is no distinction between  $t_-$  and  $t_+$ ). Thus we get two families of pseudo-holomorphic curves

$$\frac{\epsilon_a^b}{\sqrt{\det(g)}} \partial_b X^\mu \pm i J_\rho^\mu \partial_a X^\rho = 0 \quad (24)$$

where now  $J_\mu^\nu = G_{\mu\rho} \frac{\epsilon^{\rho\nu}}{\sqrt{\det(G)}}$  and  $G$  is the metric on  $M^2$ . These are the maps of [6] (here  $g_{ab}$  is the elementary metric). On this basis one can state a bold hypothesis that  $YM_4$  is localized on the rigid string instantons<sup>6</sup>. All these similarities suggest that rigid string instantons will play a significant role in string description of YM fields. Some other ideas along this line were posed in [18].

We also notice strange coincidence of the dimensions of the moduli space of genus zero rigid string instantons on  $R^4$  and  $S^4$  and the moduli space of  $SU(2)$  Yang-Mills instantons (with the appropriate identification of topological numbers). For  $k = 1$  both moduli spaces are identical. We do not know what happens for other  $k$ .

### 4.3 3d manifolds

Finally we comment on 3d target manifolds. In this case the tensor  $t^{\mu\nu}$  has 3 components. Classification of immersions of surfaces in  $R^3$  is more complicated then for the  $R^4$  case. There are  $4^h$  distinct regular homotopy classes of immersions of a surface of genus  $h$  into  $R^3$  [19]. One can easily derive appropriate instanton equations following [3] and using  $\chi$  only (the self-intersection number  $I$  is strictly 4d notion!). The equations are just Eqs.(7) with  $t^{\mu\nu}$  in place of  $t_\mu^{+\nu}$ . One of the equation is equivalent to  $\Delta X^\mu = 0$  another one represents so-called totally umbilic maps. In the case of  $\Sigma = S^2$  we have an immediate solution of the latter. This is just the sphere embedded in  $R^3$  given by Eq.(18). Unfortunately, because classification of immersions is so different and we do not know the invariant which would distinguish all topological classes it is hard to imagine that the instantons will represent all of them.

**Acknowledgment.** I would like to thank Erwin Schrödinger Institute for kind hospitality where a part of this paper was prepared. I also thank E.Corrigan, K.Gawędzki,

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<sup>6</sup>This is a natural generalization of [17].



H.Grosse, B.Jonson, C.Klimcik, A.Morozov, R.Ward and J.Zakrzewski for comments and interest in the work. Special thanks to P. Nurowski for many illuminating discussions on twistor space.

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